Multiscale Substitution Tilings

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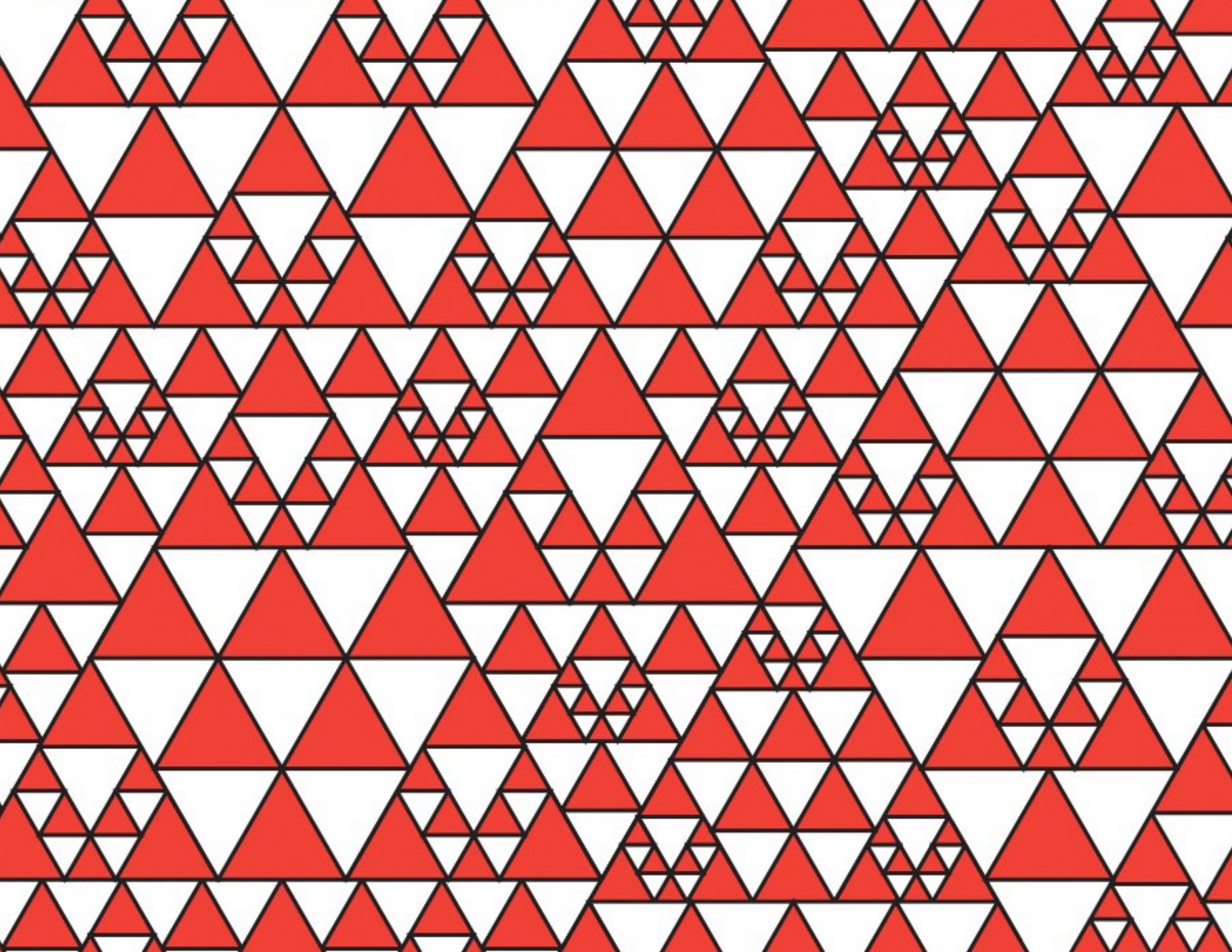
Joint work with Yaar Solomon, BGU



Multiscale Substitution Schemes

A multiscale substitution scheme $\sigma = (\tau_\sigma, \rho_\sigma)$ consists of

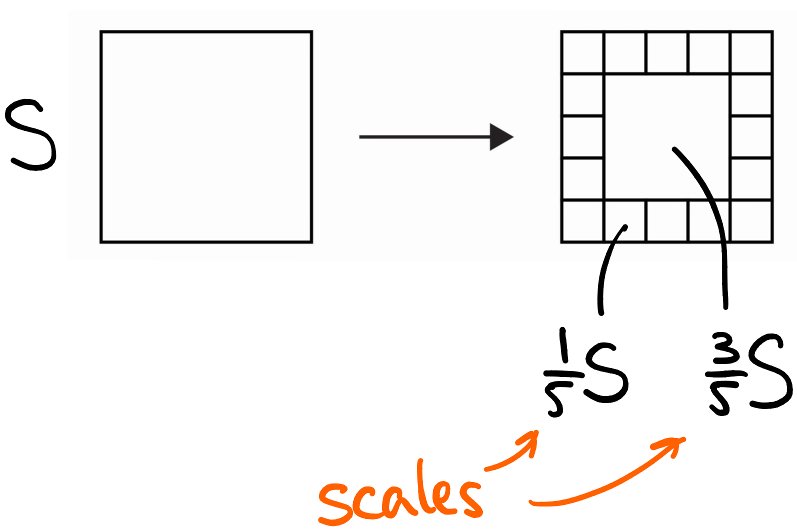
\# of prototiles

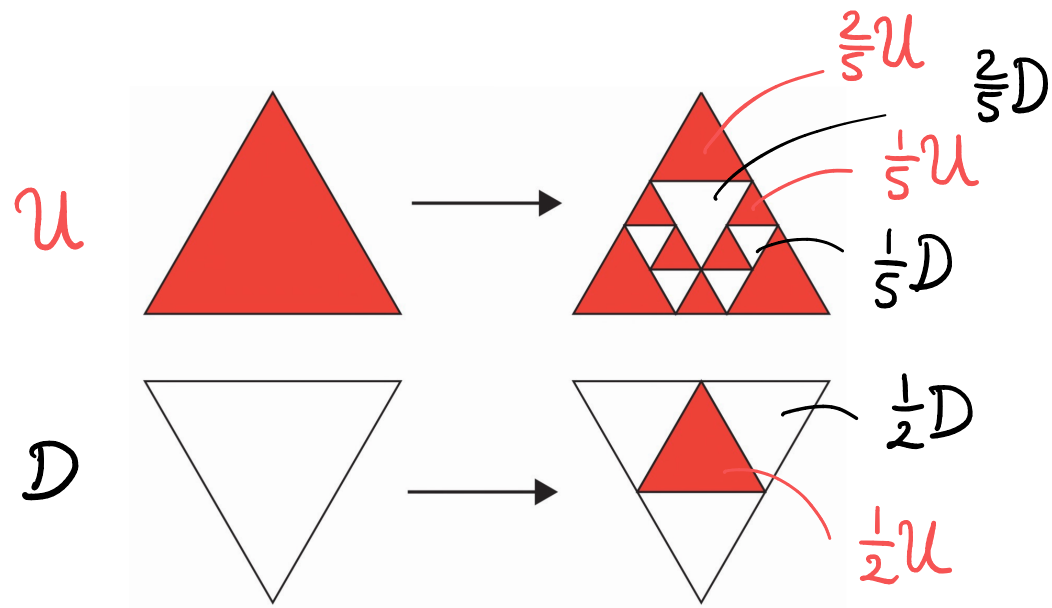
Prototiles $\tau_\sigma = (T_1, \dots, T_n)$ of unit volume in $\mathbb{R}^d$

dimension

Substitution rule ρ_σ defining a partition $\rho_\sigma(T_i)$ of each $T_i \in \tau_\sigma$ into unions of rescaled prototiles

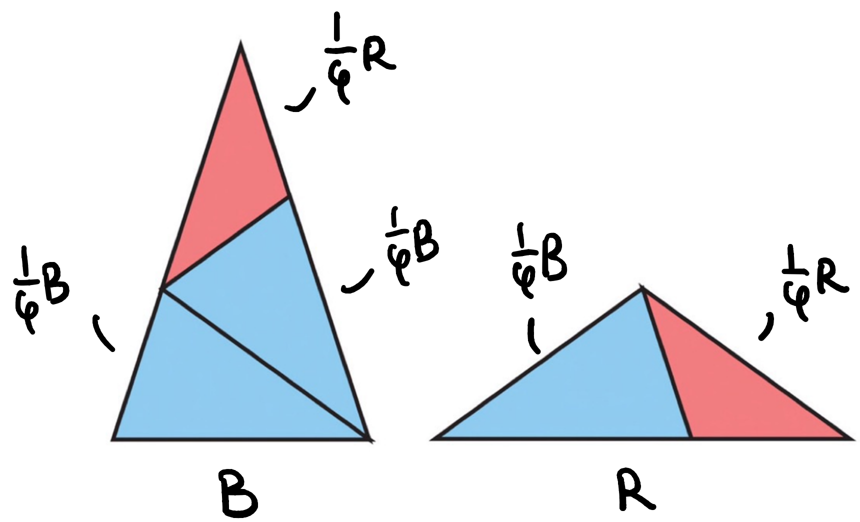
Examples:





Standard Substitution Tiling Construction

A fixed single scale λ (prototiles may vary in volume)



substitute according to substitution rule

rescale uniformly by $1/\lambda$

Every iteration defines a larger patch of tiles

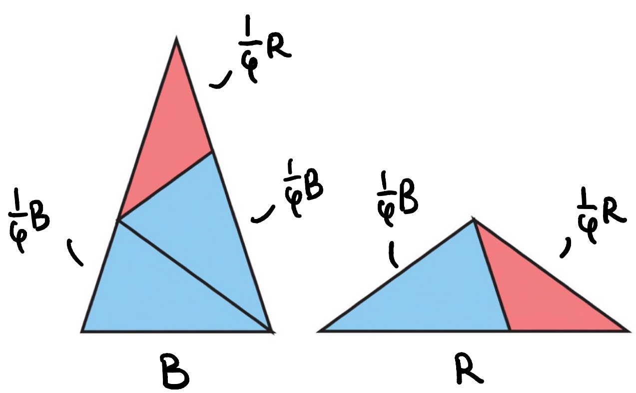
Standard Substitution Tiling Construction

Taking appropriate limits defines substitution tilings of $\mathbb{R}^d$

- Induce Delone sets (rel. dense and unif. discrete)
- Have a finite # of tiles up to translations (sometimes FLC)
- Substitution matrix $S \in M_n(\mathbb{Z})$ (irreducible, sometimes primitive)

$$S_{ij} = \#\{\text{copies of } T_j \text{ in } \rho_0(T_i)\}$$

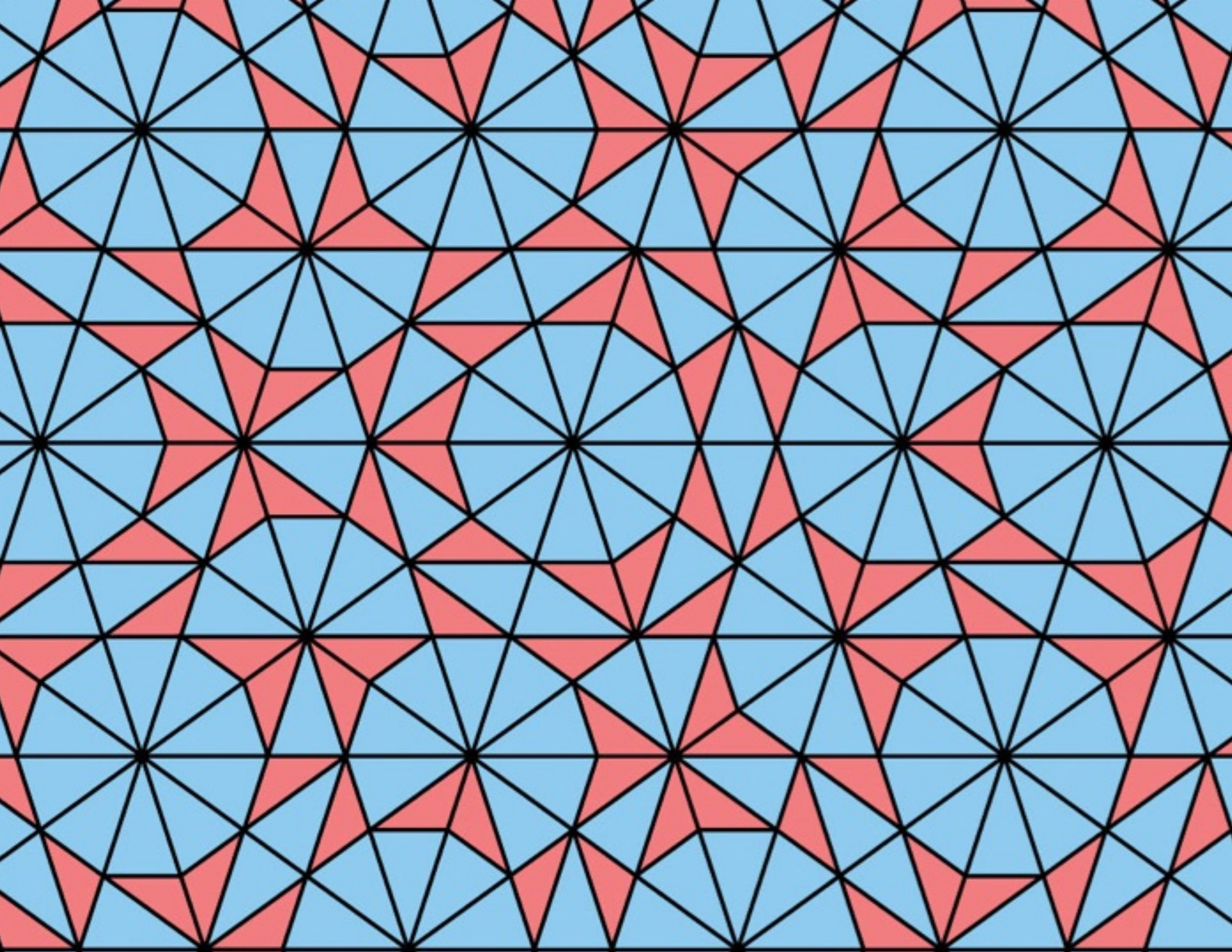
- Perron-Frobenius theory



The substitution matrix is

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Perron-Frobenius eigenvalue φ^2

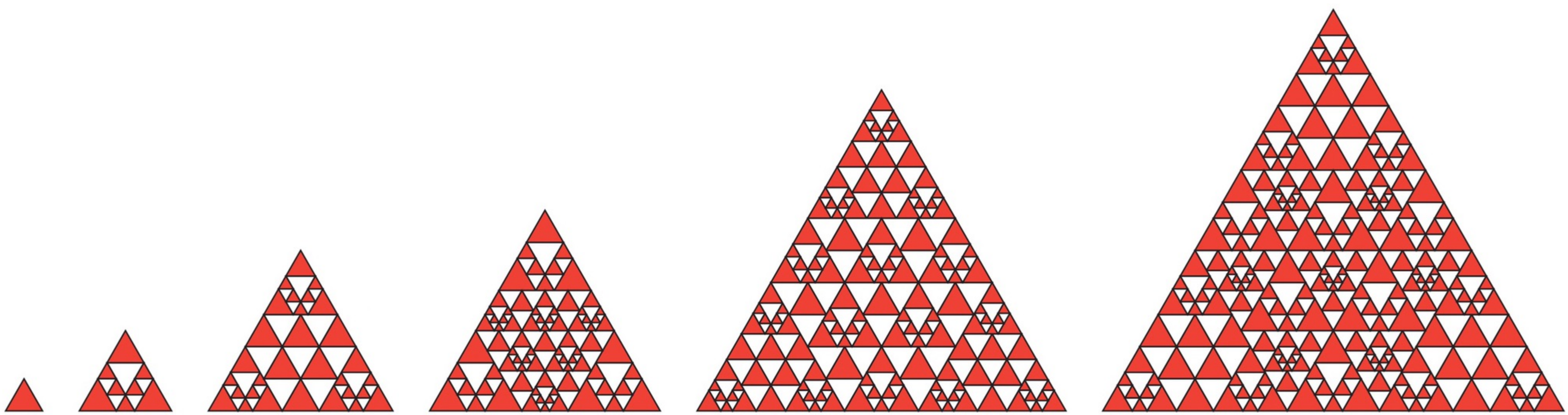


Substitution Flow

Position $T_i \in \mathcal{T}_\sigma$ so that the origin is in the interior of T_i.

Define the substitution flow $F_t(T_i)$ for $t \geq 0$ (time)

- At time $t=0$ $F_0(T_i) = T_i$ (a patch consisting of 1 tile)
- As t increases, inflate the patch by e^t and substitute tiles of volume > 1 via ρ_σ (substitution rule)



$t=0$ $t=\log 5/2$ $t=\log 5$ $t=\log 25/4$ $t=\log 10$ $t=\log 25/2$

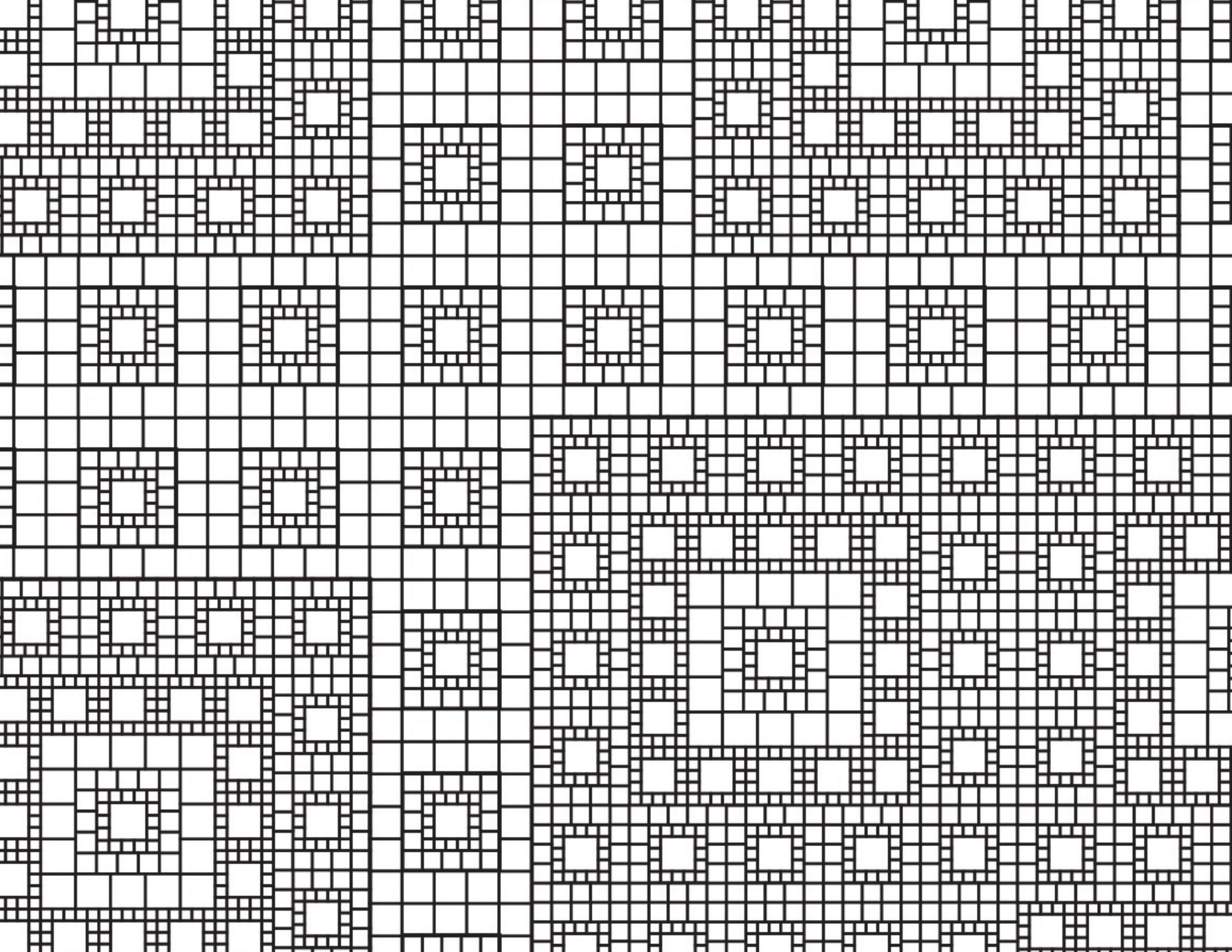
Tilings As Limits

Multiscale substitution tilings are tilings of $\mathbb{R}^d$ that are limits of translations of patches $\{F_t(T_i) : t>0, T_i \in \tau_\sigma\}$, with respect to the Chabauty–Fell topology on the space of closed subsets of $\mathbb{R}^d$, induced by the metric

$$D(A_1,A_2) = \inf\left(\left\{ r>0 : \begin{array}{l} A_1 \cap B(0,1/r) \subset A_2^{+r} \\ A_2 \cap B(0,1/r) \subset A_1^{+r} \end{array} \right\} \cup \{1\}\right)$$

"Sets are close if restricting to a large centered ball, each is contained in a small neighborhood of the other"

The tiling space X_σ consists of all tilings generated by σ.



Stationary Tilings

Choose $s \in \mathbb{R}$ and an initial position of T_i so that the patch $F_s(T_i)$ contains T_i as a tile in the same position (under the assumptions we will introduce this is possible).

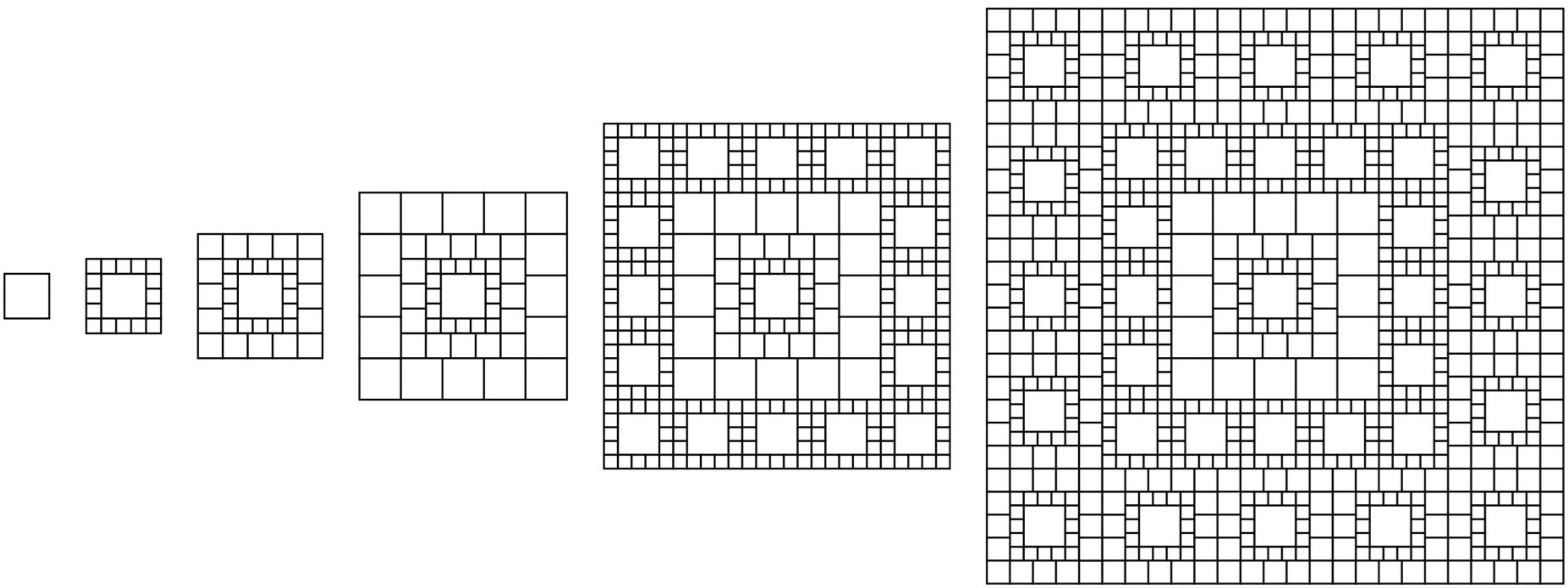
Then for all $k \in \mathbb{N}$, $F_{ks}(T_i)$ contains $F_{(k-1)s}(T_i)$, so

$$S = \bigcup_{k=0}^{\infty} F_{ks}(T_i) \in X_\sigma$$

is a stationary tiling, satisfying $F_s(S) = S$.

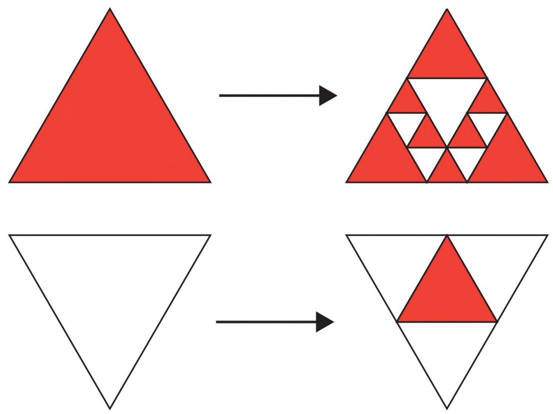
Example With Square Scheme

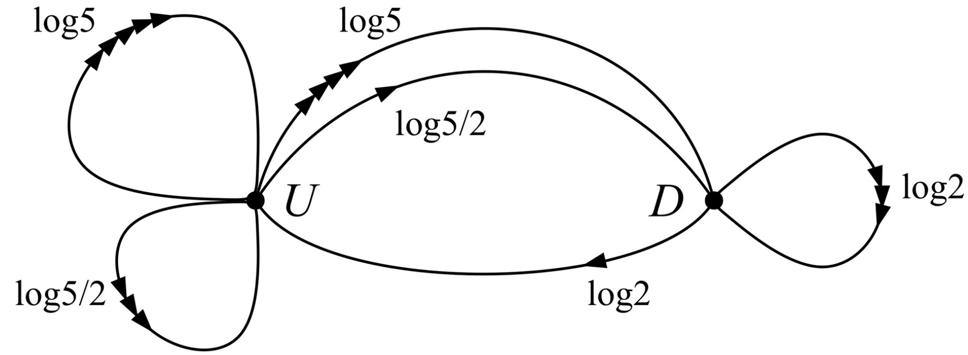
Set $s = \log \frac{5}{3}$ and consider $F_{ks}(S)$ for $k = 0, 1, \dots, 5$



Graph Model For Substitution Schemes

A directed weighted graph G_σ is associated with σ





Vertices model the prototiles in τ_σ.

Edges originating at a vertex model the tiles appearing in the substitution rule of the corresponding prototile.

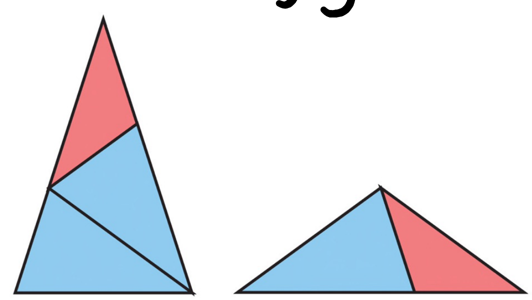
Lengths determined by the scales of the tiles $\left(\log \frac{1}{\alpha}\right)$

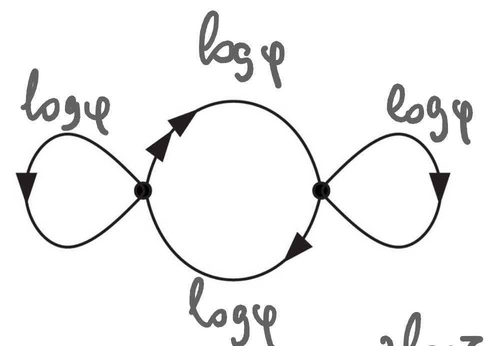
Incommensurability and Irreducibility

A substitution scheme is incommensurable if G_σ contains two closed paths of lengths a, b so that $\frac{a}{b} \notin \mathbb{Q}$.

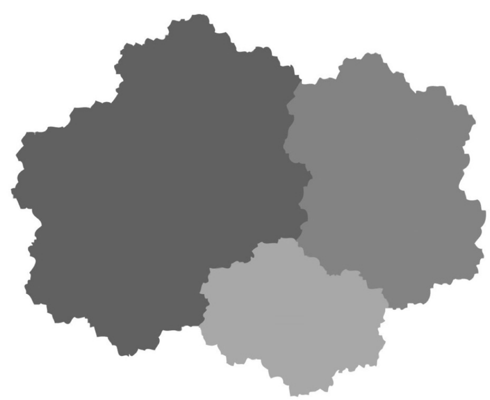
It is irreducible if G_σ is strongly connected.

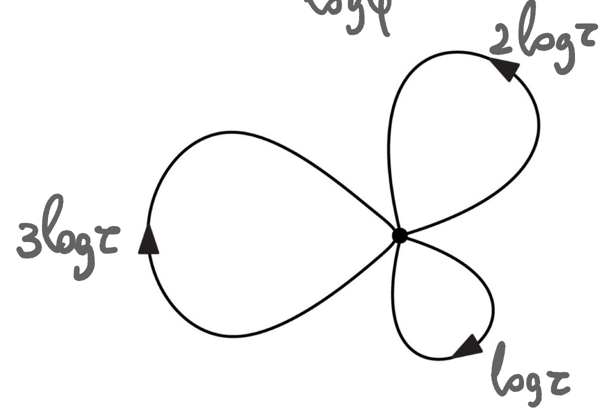
Penrose-Robinson scheme



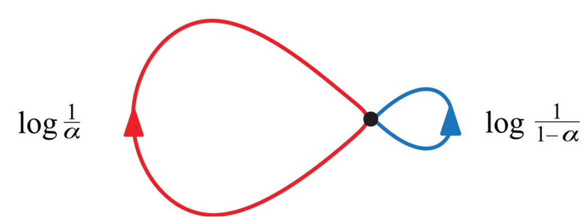


Rauzy scheme





The α-Kakutani scheme



Does This Generate New Tilings?

Yes! If we assume incommensurability. In fact

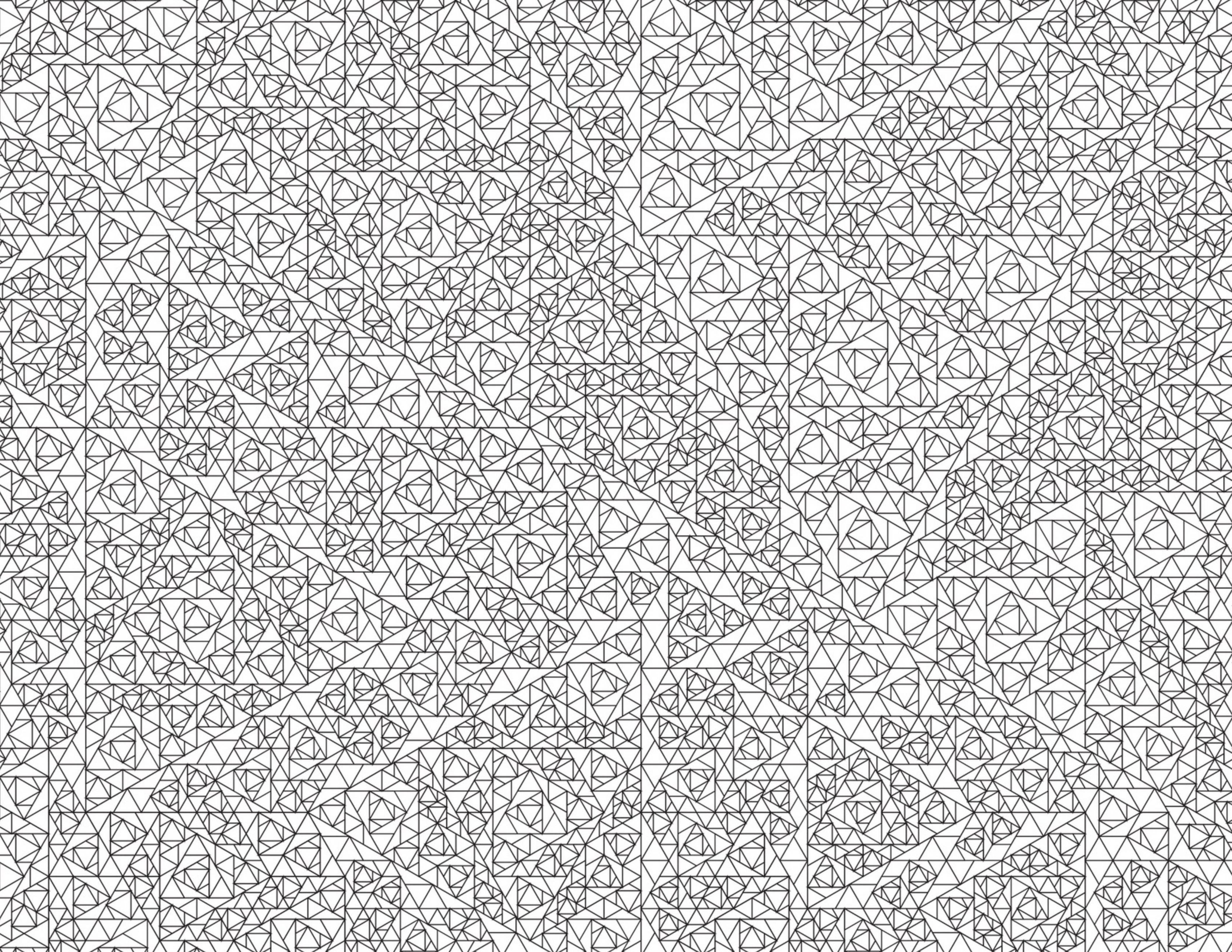
- Commensurable multiscale substitution tilings can be generated as standard substitution tilings generated by some fixed scale scheme.

- Incommensurable tilings can not.

rationals

irrationals

From now on all schemes are irreducible and incommensurable.



Tile Shape and Denseness of Scales

- For every tiling in X_σ, all tiles are similar to rescaled copies of the prototiles in τ_σ.

- Tiles appear in a dense set of scales within certain intervals of possible scales bounded away from 0.
  It follows that tilings induce Delone sets.

- The same holds for any legal patch - a subpatch of $F_t(T_i)$ for some $t \geq 0$ and $T_i \in \tau_\sigma$.

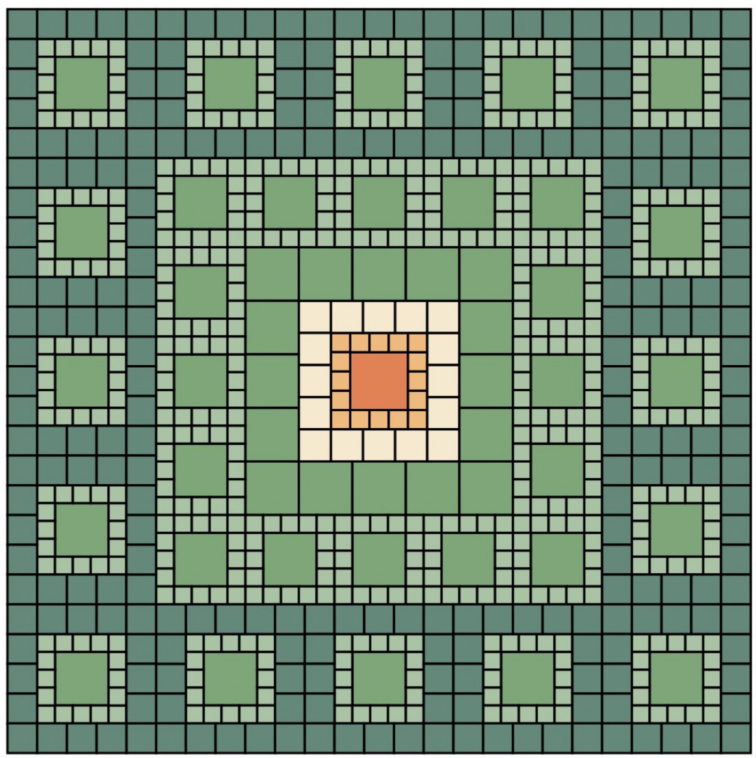
Scale Complexity Theorem

For stationary $S = \bigcup_{k=0}^{\infty} F_{ks}(T)$ the complexity function $c_S(k)$ counts the number of distinct scales in which tiles appear in $F_{ks}(T_i)$.
If $c_S(\ell) = c_S(\ell+1)$ for some $\ell \in \mathbb{N}$ then $c_S(k) = c_S(\ell)$ for all $k \geq \ell$, and such ℓ exists if and only if σ is commensurable.

A "Sturmian" tiling, in which

$c_S(k+1) = c_S(k) + 1$

for all $k \geq 0$

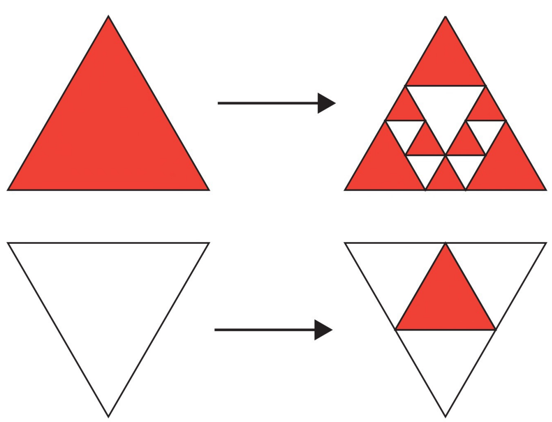


Explicit Counting Formulas

Consider the following matrices in $M_n(\mathbb{R})$ associated with σ.

Here $\sum^*$ denotes summation over all tiles T of type j in $\rho_\sigma(T_i)$

- Substitution matrix $(S_\sigma)_{ij} = \#\{\text{rescaled copies of } T_j \text{ in } \rho_\sigma(T_i)\} = \sum^* 1$ — # of reds in white
- Weighted substitution matrix $(W_\sigma)_{ij} = \sum^* \mathrm{vol}(T)$ — total red area in white
- Entropy matrix $(H_\sigma)_{ij} = \sum^* -\mathrm{vol}(T)\log(\mathrm{vol}(T))$ — contribution of reds to entropy of white



$$S_\sigma = \begin{pmatrix} 8 & 5 \\ 1 & 3 \end{pmatrix}$$

$$W_\sigma = \begin{pmatrix} \frac{17}{25} & \frac{8}{25} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \begin{cases} \text{PF eigenvalue} = 1 \\ \text{Right PF eigenvector} = \mathbb{1} = (1,1)^T \\ \text{Left PF eigenvector} = w = \left(\frac{1}{4}, \frac{8}{25}\right)^T \end{cases}$$

always true

$$H_\sigma = \begin{pmatrix} -\frac{12}{25}\log\frac{4}{25} - \frac{5}{25}\log\frac{1}{25} & -\frac{4}{25}\log\frac{4}{25} - \frac{4}{25}\log\frac{1}{25} \\ -\frac{1}{4}\log\frac{1}{4} & -\frac{3}{4}\log\frac{1}{4} \end{pmatrix}$$

Explicit Counting Formulas

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$$\#\{\text{tiles of type } r \text{ in } F_t(T)\} = \frac{[w^T(S_\sigma - W_\sigma)]_r}{w^T H_\sigma \mathbb{1}} \mathrm{vol}(F_t(T)) + o(\mathrm{vol}(F_t(T)))$$

$$\mathrm{vol}(\textstyle\bigcup \text{tiles of type } r \text{ in } F_t(T)) = \frac{[w^T H_\sigma]_r}{w^T H_\sigma \mathbb{1}} \mathrm{vol}(F_t(T)) + o(\mathrm{vol}(F_t(T)))$$

where $w \in \mathbb{R}^n$ is a left PF eigenvector of W_σ.

Distribution of Tile Scales

If in addition we define

- Density matrix $(D_\sigma(x))_{ij} = \sum^{*} g_{vol(T)}(x)$, $g_\alpha(x) = \begin{cases} \frac{\alpha}{x^{d+1}}, & \alpha < x < 1 \\ 0 \end{cases}$

$\#\{\text{tiles in } F_t(T) \text{ of type } r \text{ and scale in } [a,b]\}$

$$= \frac{d}{w^T H_\sigma \mathbb{1}} \int_a^b (w^T D_\sigma(x))_r \, dx \cdot vol(F_t(T)) + o(vol(F_t(T)))$$

Corollaries

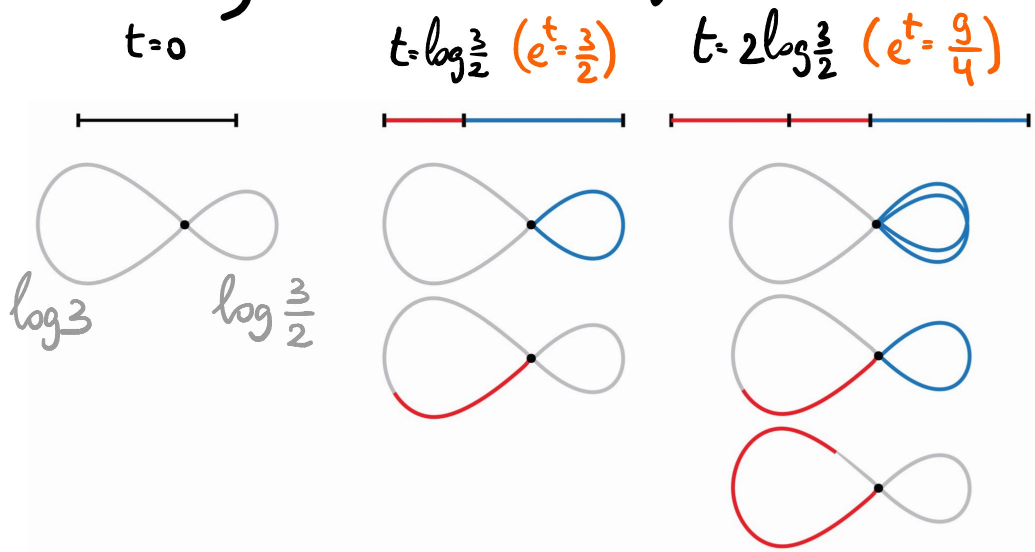
- For any α: $\#\{\text{copies of } \alpha T_r \text{ in } F_t(T)\} = o(vol(F_t(T)))$.
- Gap distribution for point sets defined as tile boundaries of 1 dimensional tilings

$$\frac{\#\{\text{neighboring points in } [-N,N] \text{ of distance in } [a,b]\}}{\#\{\text{points in } [-N,N]\}} \to \frac{1}{w^T(S_\sigma - W_\sigma)\mathbb{1}} \int_a^b w^T D_\sigma(x) \mathbb{1} \, dx$$

Counting: The Substitution Flow and the Graph

- Every tile in $F_t(T_i)$ corresponds to a unique metric path of length t in G_σ which originates at vertex i.
- If the tile is a copy of αT_j, then the path terminates on an edge terminating at vertex j, at distance $\log \frac{1}{\alpha}$ from j.

$\frac{1}{3}$- Kakutani:



- Counting is translated to problems on the distribution of paths on incommensurable graphs [Kiro, Smilansky x2, 2020]

Bounded Displacement Equivalence

- Delone sets Λ, Γ are BD equivalent if $\exists$ bijection $\varphi: \Lambda \to \Gamma$ with $\sup_{x \in \Lambda} \| x - \varphi(x) \| < \infty$ (BD map).
  A tiling is uniformly spread if $\exists$ Λ, obtained by picking a point from each tile, which is BD equivalent to a lattice.
- Laczkovich criterion: being uniformly spread is equivalent to sufficiently bad tile counting error terms.
- Tilings in X_σ are never uniformly spread.
- Moreover, the set of BD equivalence classes represented in X_σ has cardinality of the continuum [S', Solomon].

Uniform Patch Frequencies Theorem

Tilings $\mathcal{T} \in X_\sigma$ have uniform patch frequencies:

For any legal patch P in $\mathcal{T}$ and a bounded interval I that contains a left neighborhood of 1:

$\alpha \in I$

$$\lim_{q \to \infty} \frac{\#\text{ appearances of } \alpha P \text{ in } \mathcal{T} \text{ inside } A_q + h}{\mathrm{vol}(A_q)} =: \mathrm{freq}(P, I, \mathcal{T})$$

exists uniformly in $h \in \mathbb{R}^d$, is positive and independent of $\mathcal{T} \in X_\sigma$ and of a choice of a van Hove sequence (A_q) in $\mathbb{R}^d$.

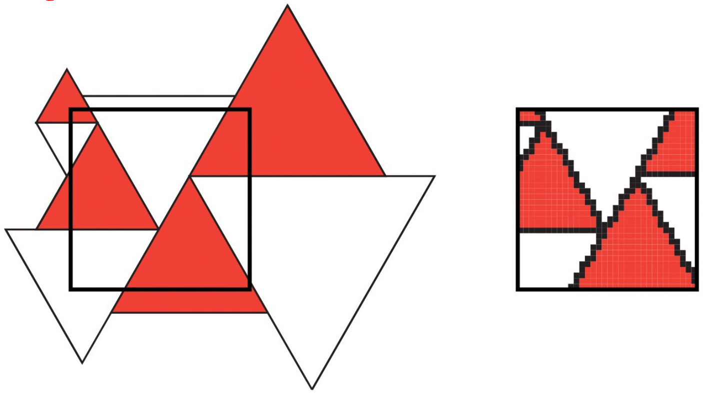
(A sequence of bounded measurable sets $\lim_{q \to \infty} \frac{\mathrm{vol}\left((\partial A_q)^{+r}\right)}{\mathrm{vol}(A_q)} = 0$)

Dynamics Theorems

- X_σ is invariant under translations in $\mathbb{R}^d$ and under F_t.
  $F_t(\mathcal{T}-x) = F_t(\mathcal{T}) - e^t x$ for $t \geq 0$, $x \in \mathbb{R}^d$ (horospheric and geodesic).
- Periodic orbits of F_t correspond to stationary tilings.
- The dynamical system $(X_\sigma, \mathbb{R}^d)$ is minimal.
  - Tilings $\mathcal{T} \in X_\sigma$ are almost repetitive (relative denseness of return times to ε-neighborhoods of patches, for all $\varepsilon > 0$).
  - Every $\mathcal{T}_1, \mathcal{T}_2 \in X_\sigma$ are almost locally indistinguishable (same ε-neighborhoods up to translations, for all $\varepsilon > 0$).
- $(X_\sigma, \mathbb{R}^d)$ is uniquely ergodic.

Steps of Proof of Unique Ergodicity

- Cylinder sets are defined according to "pixelizations"

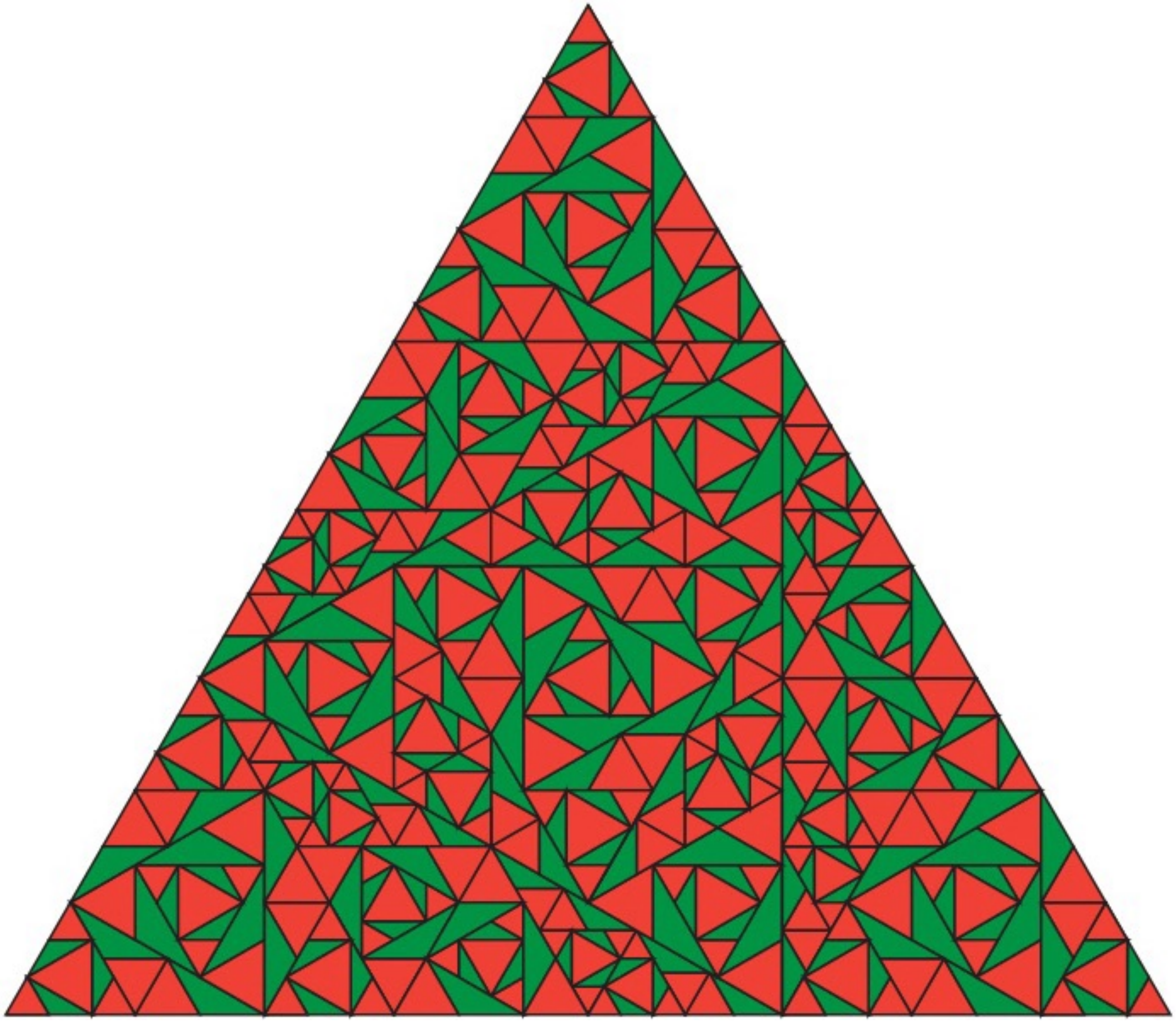


A cylinder set consists of all tilings with a pixelization compatible with a fixed coloring of a large cube

- The ergodic average $\frac{1}{vol(A_q)} \int_{A_q} \chi_c(S-x-h)dx$ converges to a countable sum of counting functions of appearances of patches in A_q+h.
- Denseness of scales in which patches appear allow the interpretation of the sum as a Riemann-Stieltjes integral, allowing an evaluation using the uniform patch frequencies.

Short Summary of New Tools

- substitution matrix → substitution graph
- discrete-time substitution and inflation → substitution flow
- primitivity → incommensurability
- Perron-Frobenius → distribution of paths on weighted graphs
- uniform patch frequency → UPF with scale intervals
- summation on patches → Riemann-Stieltjes integration



Thanks!